No collaboration or aids are allowed. **Prove every statement.** Feel free to cite standard facts without proof, but clearly state the results you are using. If you write a partial solution, clearly indicate where the gaps are.

Write your answers as **legibly** as you can on the blank sheets of paper provided. Write **complete** answers in **complete sentences**. Make sure that your **notation is defined**!

Use only **one side** of each sheet; start each problem on a **new sheet** of paper; and be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.

If you believe a problem is incorrect, then give a counterexample and/or supply the missing hypothesis and prove the resulting statement. If some problem is vague, then be sure to explain your interpretation of the problem.

Each problem is worth 10 points.

Problem 1 Suppose p and $p+2$ are primes. Classify groups of order $p^3 + 2p^2$ up to isomorphism.

Problem 2 Is the following statement true or false? "If H and K are normal subgroups of a finite group G, with $H \cong K$, then $G/H \cong G/K$."

Problem 3 Let G be a group of order p^n for some prime p and let H be a normal subgroup of G, with $H \neq \{1\}$. Prove that $Z(G) \cap H \neq \{1\}$, where $Z(G)$ is the center of G.

Problem 4 Give an example of a non-zero prime ideal that is not a maximal ideal. Prove that if I is a non-zero prime ideal in a Principal Ideal Domain, then I is a maximal ideal.

Problem 5 Recall that a square matrix M is nilpotent if $M^n = 0$ for some positive integer n. Let M_1 and M_2 be 6×6 nilpotent matrices over the field of complex numbers $\mathbb{C}$. Suppose that M_1 and M_2 have the same minimal polynomial and the same nullity. Prove that M_1 and M_2 are similar. Show that this is not necessarily true for 7×7 nilpotent matrices.

Problem 6 Let R be the polynomial ring $R = k[x, y]$, where k is a field, and let M be the R-module

$$M := \left\{ \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} \in R^3 \;\middle|\; r_1x^3 + r_2x^2y + r_3y^2 = 0 \right\}.$$

Find a finite generating set for M.

Problem 7 Let R be a commutative ring. For $r \in R$, the *annihilator of* r, defined by

$$\operatorname{ann}(r) := \{s \in R \mid sr = 0\} ,$$

is an ideal of R (you do not need to prove this). Define $X := \{\operatorname{ann}(r) \mid r \in R \setminus \{0\}\}$. Show that the maximal elements of X are prime ideals of R.

Problem 8 Let k be a field and R be a subring of k. For $\alpha \in k$, recall that $R[\alpha]$ denotes the minimal R-subalgebra of k containing both R and α. Let

$$A := \{\alpha \in k \mid R[\alpha] \text{ is finitely generated as an } R\text{-module}\}.$$

Prove that A is a ring.

Problem 9 Determine the number of monic irreducible polynomials of degree 5 in the polynomial ring $\mathbb{F}_{11}[x]$, where $\mathbb{F}_{11}$ is the field of 11 elements.

Problem 10 For a positive integer n, let $\alpha = \sum_{k=1}^{n} \sqrt{k}$. Prove that the minimal polynomial of α over $\mathbb{Q}$ has degree $2^{\pi(n)}$ where $\pi(n)$ is the number of prime numbers $\leq n$.